PROPERTIES OF REAL 2 × 2 ORTHOGONAL MATRICES
AND THEIR RELATIONSHIP WITH PLANE ROTATIONS

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Abstract. We begin by partitioning all real 2 × 2 orthogonal matrices into two forms: symmetric and asymmetric. Certain known properties of these matrices are stated. Properties regulating sums and products of these matrices are then developed. Symmetric and asymmetric plane rotations are then defined. Since all real orthogonal matrices can be represented as a product of plane rotations, all properties given previously for 2 × 2 orthogonal matrices are extended to plane rotations. Five tables of examples are given.

1. Introduction. Orthogonal matrices are fundamental in the areas of linear models and experimental design. They are highly used in linear algebra and other related fields of pure and applied mathematics.

Products of orthogonal matrices are often used to reduce matrices to upper or lower triangular form or to bidiagonal or tridiagonal form. Products of elementary reflections and plane rotations are types of orthogonal matrices which have been found to be particularly useful in this regard. In this paper we investigate plane rotations. Products of 2 × 2 orthogonal matrices are basic to this study.

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2. Real Two by Two Orthogonal Matrices. One can easily verify that all real 2×2 orthogonal matrices can be represented in either the symmetric orthogonal form

$$S(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = S(\theta + 2\pi k)$$

or the asymmetric orthogonal form

$$A(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A(\theta + 2\pi k)$$

where $0 \leq \theta < 2\pi$ and k is any integer.

Further, easily verified properties include:

Property 1

(i) The eigenvalues of $S(\theta)$ are 1 and -1.

(ii) The eigenvalues of $A(\theta)$ are $\cos\theta \pm i \sin\theta$.

Property 2

(i) $|S(\theta)| = -1$

(ii) $|A(\theta)| = 1$

Now we are ready to state and prove properties on sums and products of 2×2 orthogonal matrices.

Property 3

$\sum_{i=1}^{n} \lambda_i S(\theta_i)$ is of the form $S(\theta)$ and $\sum_{i=1}^{n} \lambda_i A(\theta_i)$ is of the form $A(\theta)$

iff $\sum_{i=1}^{n} \lambda_i^2 + \sum_{i \neq j}\sum \lambda_i \lambda_j \cos(\theta_j - \theta_i) = 1.$

<u>Proof</u>: $$\sum_{i=1}^{n} \lambda_i S(\theta_i) = \begin{bmatrix} \sum_{i=1}^{n} \lambda_i \cos\,\theta_i & \sum_{i=1}^{n} \lambda_i \sin\,\theta_i \\ \sum_{i=1}^{n} \lambda_i \sin\,\theta_i & -\sum_{i=1}^{n} \lambda_i \cos\,\theta_i \end{bmatrix}$$

which is of the form $S(\theta)$ iff

$$\left(\sum_{i=1}^{n} \lambda_i \cos\,\theta_i\right)^2 + \left(\sum_{i=1}^{n} \lambda_i \sin\,\theta_i\right)^2 = 1$$

but using $\sin^2\,\theta_i + \cos^2\,\theta_i = 1$ and $\cos(\theta_j - \theta_i) = \cos\,\theta_j\,\cos\,\theta_i + \sin\,\theta_j\,\sin\,\theta_i$ we have

$$\sum_{i=1}^{n} \lambda_i^2 + \sum_{i\neq j}\sum \lambda_i \lambda_j\,\cos(\theta_j - \theta_i) = 1.$$

The proof for $\sum_{i=1}^{n} \lambda_i A(\theta_i)$ is nearly identical.

In the special case of Property 3 where $\theta_i = \theta$ for all i

$$\sum_{i=1}^{n} \lambda_i^2 + \sum_{i\neq j}\sum \lambda_i \lambda_j\,\cos(\theta_j - \theta_i) = 1$$

reduces to

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 = 1$$

or

$$\sum \lambda_i = \pm 1$$

so that

$$\sum_{i=1}^{n} \lambda_i S(\theta) = \pm S(\theta)$$

$$\sum_{i=1}^{n} \lambda_i A(\theta) = \pm A(\theta)$$

which are the obvious results.

Property 4

$$\text{(i)} \quad \prod_{i=1}^{n} A(\theta_i) = A\left(\sum_{i=1}^{n} \theta_i\right) = A(\varphi)$$

where $\sum_{i=1}^{n} \theta_i = \varphi + 2\pi k$ and $0 \le \varphi < 2\pi$

$$\text{(ii)} \quad \prod_{i=1}^{n} S(\theta_i) = \begin{cases} S(\alpha) = S(\phi) & \text{if } n = 2k + 1 \\ A(\alpha) = A(\phi) & \text{if } n = 2k \end{cases}$$

where $\alpha = \sum_{i=1}^{n} (-1)^{n-i} \theta_i = \phi + 2\pi k$ and $0 \le \phi < 2\pi$

Proof: For $0 \le \theta_1, \theta_2 < 2\pi$ we have

$$A(\theta_1)A(\theta_2) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = A(\theta_1 + \theta_2) = A(\alpha_1)$$

where $\theta_1 + \theta_2 = \alpha_1 + 2\pi k$ and $0 \le \alpha_1 < 2\pi$. Consequently, repeating we have

$$A(\theta_1)A(\theta_2)A(\theta_3) = A(\theta_1 + \theta_2)A(\theta_3) = A(\theta_1 + \theta_2 + \theta_3)$$
$$\vdots$$
$$\prod_{i=1}^{n} A(\theta_i) = A\left(\sum_{i=1}^{n} \theta_i\right) = A(\varphi)$$

where $\sum_{i=1}^{n} \theta_i = \varphi + 2\pi k$ and $0 \le \varphi < 2\pi$.

Now

$$S(\theta_1)S(\theta_2) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_2 - \theta_1) & \sin(\theta_2 - \theta_1) \\ -\sin(\theta_2 - \theta_1) & \cos(\theta_2 - \theta_1) \end{bmatrix} = A(\theta_2 - \theta_1)$$

and

$$S(\theta_1)A(\theta_2) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \sin\theta_1 & -\cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & -\cos(\theta_1 + \theta_2) \end{bmatrix} = S(\theta_1 + \theta_2)\text{1/}.$$

1/ In a similar manner, we obtain $A(\theta_1)S(\theta_2) = S(\theta_2 - \theta_1)$.

So if n is odd

$$\prod_{i=1}^{n} S(\theta_i) = S(\theta_1)A(\theta_3 - \theta_2)A(\theta_5 - \theta_4) \cdots A(\theta_n - \theta_{n-1})$$

$$= S(\theta_1 - \theta_2 + \theta_3)A(\theta_5 - \theta_4) \cdots A(\theta_n - \theta_{n-1})$$

$$= \cdots = S(\theta_1 - \theta_2 + \theta_3 \cdots \theta_n) = S(\alpha) = S(\phi).$$

If n is even

$$\prod_{i=1}^{n} S(\theta_i) = A(\theta_2 - \theta_1)A(\theta_4 - \theta_3)A(\theta_6 - \theta_5) \cdots A(\theta_n - \theta_{n-1})$$

$$= A(\theta_4 - \theta_3 + \theta_2 - \theta_1)A(\theta_6 - \theta_5) \cdots A(\theta_n - \theta_{n-1})$$

$$= \cdots = A(\alpha) = A(\phi)$$

where $\alpha = \sum_{i=1}^{n} (-1)^{n-i}\theta_i$ and $\alpha = \phi + 2\pi k$ where $0 \leq \phi < 2\pi$.

Corollary to Property 4

(i) $S(\theta_1)S(\theta_2) = A(\theta_2 - \theta_1)$

(ii) $S(\theta_1)A(\theta_2) = S(\theta_1 + \theta_2)$

(iii) $A(\theta_1)S(\theta_2) = S(\theta_2 - \theta_1)$

(iv) $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$

Proof: Proved in proof of Property 4.

In view of Property 4, one could equivalently call $S(\theta)[A(\theta)]$ the odd [even] product form of 2×2 orthogonal matrices.

In this paper a $p \times p$ matrix R will be called an elementary reflector (ER) if R can be written as

$$R = I - 2x(x'x)^{-1}x' \text{ (or } I - 2ww'\text{)}$$

where x is a non-zero p $\times$ 1 vector of real components (and $w = x/\|x\|$). (See [1], pages 139-144; [2], pages 230-244; [3], pages 48-50.) Also, for $w = (\sin \phi/2, -\cos \phi/2)$ we have

$$S(\theta) = I - 2ww'$$

so $S(\theta)$ is an elementary reflector (Householder matrix) and all 2 $\times$ 2 elementary reflectors are of the form $S(\theta)$. Consequently, the set of 2 $\times$ 2 elementary reflectors is identical to the set of 2 $\times$ 2 symmetric orthogonal matrices.

3. <u>Symmetric and Asymmetric Plane Rotations</u>. A symmetric plane rotation is merely a modification of the identity matrix by replacing certain elements by the elements of a symmetric 2 $\times$ 2 orthogonal matrix (an elementary reflector or Householder matrix). Thus we will denote a symmetric plane rotation by

$$I_S(\theta) = I_S^n(\theta^{i,j}) = \begin{array}{c@{}c} & \begin{array}{cccccc} & i & & j & & n \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ n \end{array} & \left[\begin{array}{cccccc} 1 \ddots 1 & \vdots & & \vdots & & \bigcirc \\ \cdots & \cos\theta & & \sin\theta & & \\ & & 1 \ddots 1 & & & \\ \cdots & \sin\theta & & -\cos\theta & & \\ \bigcirc & & & & 1 \ddots 1 & \end{array} \right] \end{array}$$

Similarly, an asymmetric plane rotation is merely a modification of the identity matrix by replacing certain elements by the elements of an asymmetric 2 $\times$ 2 orthogonal matrix. Consequently, we will denote an asymmetric plane rotation by

$$I_A(\theta) = I_A^n(\theta^{i,j}) = \begin{array}{c@{}c} & \begin{array}{ccccccc} & & & i & & j & n \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ n \end{array} & \begin{bmatrix} 1 & & & \vdots & & \vdots & \bigcirc \\ & \ddots & & \vdots & & \vdots & \\ & & 1 & \vdots & & \vdots & \\ \cdots & \cdots & & \cos\theta & & \sin\theta & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ \cdots & \cdots & & -\sin\theta & & \cos\theta & \\ & & & & & 1 & \\ & \bigcirc & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \end{array}$$

Equations (43.1) and (43.2) on page 47 of reference [3] is a type of elementary unitary matrix which is a generalization of symmetric and asymmetric plane rotations to include complex elements. Equations (43.3) and (43.4) on the same page of this reference also define a generalization of asymmetric plane rotations to include complex elements.

The term plane rotation will be used as a general term to refer to a matrix of either the symmetric or asymmetric type. By convention, the dimension will be n and $i < j$ throughout. The superscripts i and j denote the index pair for which rotation or rotation and reflection occur. When absent, the dimension n and the index pair (i,j) is implied. The real argument θ refers to the angle of rotation when we are concerned with asymmetric matrices and the angle of rotation before or after reflection in the case of symmetric matrices. If several arguments appear this implies a new, possibly different, angle of rotation (or angle of rotation before or after reflection) for each index pair provided all indices are distinct. In particular let us extend our notation to:

<u>Definition</u>. Let $\underline{B} = (B_1, \cdots, B_k)$ be an n tuple of letters chosen from the set $\{S,A\}$ and let $r_1, r_2, \cdots, r_{2k}$ be distinct integers between 1 and n inclusive,

then

$$I_{\underline{B}}\left(\theta_1^{r_1,r_2},\ \theta_2^{r_3,r_4},\ \ldots,\ \theta_k^{r_{2k-1},r_{2k}}\right) = \prod_{\ell=1}^{k} I_{B_\ell}\left(\theta_\ell^{r_{2\ell-1},r_{2\ell}}\right) .$$

If all the components of $\underline{B}$ are of the same letter type, we shall use the abbreviated convention of placing only the letter type itself as subscript. Thus, for example,

$$I_S\left(\theta_1^{r_1,r_2},\ \theta_2^{r_3,r_4},\ \theta_3^{r_5,r_6}\right) = I_{S,S,S}\left(\theta_1^{r_1,r_2},\ \theta_2^{r_3,r_4},\ \theta_3^{r_5,r_6}\right) .$$

Plane rotations can be used to reduce an $n \times p$ rectangular matrix to an upper triangular matrix using approximately $2p(p + 1)$ multiplications per row. This method is most advantageous in the case of the availability of minimal computer storage since space need be allocated for only one row of the matrix at a time.

Properties 1 through 4 can be modified to deal with these matrices as follows:

Property 1'

(i) The eigenvalues of $I_S(\theta)$ are $n - 1$ one's and one negative one.

(ii) The eigenvalues of $I_A(\theta)$ are $n - 2$ one's and $\cos\theta + i \sin\theta$ and its conjugate.

Property 2'

(i) $|I_S(\theta)| = -1$

(ii) $|I_A(\theta)| = 1$

Property 3'

$\sum_{k=1}^{m} \lambda_k I_S(\theta_k)$ is of the form $I_S(\theta)$ and $\sum_{k=1}^{m} \lambda_k I_A(\theta_k)$ if of the form $I_A(\theta)$ iff

(i) $\sum_{k=1}^{m} \lambda_k = 1$

and

(ii) $\sum_{k=1}^{m} \lambda_k^2 + \sum_{k \neq \ell}\sum \lambda_k \lambda_\ell \cos(\theta_\ell - \theta_k) = 1.$

Proof: It follows easily upon reviewing the proof of Property 3 and the definitions of $I_S(\theta_k)$ and $I_A(\theta_k)$.

Property 3"

If either $\lambda_k > 0$ for k - 1, ..., m or $\lambda_k < 0$ for k = 1, ..., m, then $\sum_{k=1}^{m} \lambda_k I_S(\theta_k)$ is of the form $I_S(\theta)$ and $\sum_{k=1}^{m} \lambda_k I_A(\theta_k)$ is of the form $I_A(\theta)$ iff

(i') $\sum_{k=1}^{m} \lambda_k = 1$

and

(ii') $\theta_k = \theta$ for all k.

Proof: From Property 3', when no conditions are imposed on the λ_k's, $\sum_{k=1}^{m} \lambda_k I_S(\theta_k)$ is of the form $I_S(\theta)$ and $\sum_{k=1}^{m} \lambda_k I_A(\theta_k)$ is of the form $I_A(\theta)$ iff

(i) $\sum_{k=1}^{m} \lambda_k = 1$

and

(ii) $\sum_{k=1}^{m} \lambda_k^2 + \sum_{k \neq \ell}\sum \lambda_k \lambda_\ell \cos(\theta_\ell - \theta_k) = 1.$

Upon squaring both sides of (i), we have

$$\left(\sum_{k=1}^{m} \lambda_k\right)^2 = \sum \lambda_k^2 + \sum_{k \neq \ell}\sum \lambda_k \lambda_\ell = 1.$$

Consequently, if (i) and (ii) are to be consistent, we must have

$$\sum_{k \neq \ell}\sum \lambda_k \lambda_\ell \cos(\theta_i - \theta_k) = \sum_{k \neq \ell}\sum \lambda_k \lambda_\ell \, .$$

But under the more restrictive hypotheses of Property 3", we also have the condition that either $\lambda_k > 0$ for all k or $\lambda_k < 0$ for all k. Since the products $\lambda_k \lambda_\ell$ $(k \neq \ell)$ are always nonnegative under this condition, we must have that

$$\cos(\theta_\ell - \theta_k) = 1$$

for all $\ell \neq k$. But since $0 \leq \theta_k, \theta_\ell < 2\pi$, this implies that all the θ_k's are equal. Consequently, under the restricted conditions of Property 3", (i) and (ii) reduce to

(i') $\sum_{k=1}^{m} \lambda_k = 1$

and

(ii') $\theta_k = \theta$ for all k.

Property 4'

$$\text{(i)} \quad \prod_{\ell=1}^{n} I_A(\theta_\ell) = I_A\Big(\sum_{\ell=1}^{n} \theta_\ell\Big) = I_A(\varphi)$$

where $\sum_{\ell=1}^{n} \theta_\ell = \varphi + 2\pi k$ and $0 \le \varphi < 2\pi$

$$\text{(ii)} \quad \prod_{\ell=1}^{n} I_S(\theta_\ell) = \begin{cases} I_S(\alpha) = I_S(\phi) & \text{if } n = 2k + 1 \\ I_A(\alpha) = I_A(\phi) & \text{if } n = 2k \end{cases}$$

where $\alpha = \sum_{\ell=1}^{n} (-1)^{n-\ell}\theta_\ell = \phi + 2\pi k$ and $0 \le \phi < 2\pi$

Corollary to Property 4'

$$\text{(i)} \quad I_S(\theta_1)I_S(\theta_2) = I_A(\theta_2 - \theta_1)$$

$$\text{(ii)} \quad I_S(\theta_1)I_A(\theta_2) = I_S(\theta_1 + \theta_2)$$

$$\text{(iii)} \quad I_A(\theta_1)I_S(\theta_2) = I_S(\theta_2 - \theta_1)$$

$$\text{(iv)} \quad I_A(\theta_1)I_A(\theta_2) = I_A(\theta_1 + \theta_2)$$

It should also be noted that the product of two plane rotations whose indices match exactly once will produce a matrix which is a modification of the identity matrix by replacing certain elements with the elements of a 3×3 orthogonal matrix. For example, for $j < j'$

$$
I_S(\theta_1^{i,j})I_S(\theta_2^{i,j'}) =
\begin{array}{cc}
 & \begin{array}{cccccccc} & i & & j & & j' & & n \end{array}\\
\begin{array}{c} \\ \\ i \\ \\ j \\ \\ j' \\ \\ n \end{array} &
\begin{bmatrix}
1 \ddots & \vdots & & \vdots & & \vdots & & \\
& 1 & & & & & & \\
\cdots & \cos\theta_1 \cos\theta_2 & & \sin\theta_1 & & \cos\theta_1 \sin\theta_2 & & \\
& & 1 \ddots 1 & & & & & \\
\cdots & \sin\theta_1 \cos\theta_2 & & -\cos\theta_1 & & \sin\theta_1 \sin\theta_2 & & \\
& & & & 1 \ddots 1 & & & \\
\cdots & \sin\theta_2 & & 0 & & -\cos\theta_2 & & \\
& & & & & & 1 \ddots & \\
& & & & & & & 1
\end{bmatrix}^{2/}
\end{array}
$$

Continuing in this way, a product of plane rotations with the appropriate matching indices can produce any orthogonal matrix.

2/ To make the matrix more appealing to the eye, only the non-zero elements of this matrix are explicitly given in the array.

EXAMPLES

Note: In all of the following examples, S can be replaced by I_S and A by I_A so as to extend each from 2 × 2 orthogonal matrices to plane rotations. Replacing θ with $-\theta$ in either $S(\theta)$ or $A(\theta)$ has the effect of changing the off-diagonal signs; replacing θ by $\pi - \theta$ has the effect of changing the diagonal signs. Also $S(\pi + \theta) = -S(\theta)$ and $A(\pi + \theta) = -A(\theta)$.

I. SYMMETRIC BY SYMMETRIC

$S(\theta)S(\theta) = A(0) = I$

$S(\theta)S(-\theta) = A(-2\theta)$

$S(-\theta)S(\theta) = A(2\theta)$

$S(\theta)S(\pi - \theta) = A(\pi - 2\theta)$

$S(\pi - \theta)S(\theta) = A(2\theta - \pi)$

$S(-\theta)S(\pi - \theta) = A(\pi) = -I$

$S(\pi - \theta)S(-\theta) = A(-\pi) = -I$

In general, we have

$S(\theta_1)S(\theta_2) = S(-\theta_2)S(-\theta_1) = A(\theta_2 - \theta_1)$

$S(\theta_2)S(\theta_1) = A(\theta_1 - \theta_2)$.

Thus, $S(\theta_1)S(\theta_2) \neq S(\theta_2)S(\theta_1)$ and multiplication of 2 × 2 symmetric orthogonal matrices by 2 × 2 symmetric orthogonal matrices is not commutative.

II. ASYMMETRIC BY ASYMMETRIC

$A(\theta)A(\theta) = A(2\theta)$

$A(-\theta)A(-\theta) = A(\pi - \theta)A(\pi - \theta) = A(-2\theta)$

$A(\theta)A(-\theta) = A(-\theta)A(\theta) = A(0) = I$

$A(\theta)A(\pi - \theta) = A(\pi - \theta)A(\theta) = A(\pi) = -I$

$A(-\theta)A(\pi - \theta) = A(\pi - \theta)A(-\theta) = A(\pi - 2\theta)$

In general, we have

$$A(\theta_1)A(\theta_2) = A(\theta_2)A(\theta_1) = A(\theta_1 + \theta_2).$$

Thus, multiplication of 2 X 2 asymmetric orthogonal matrices by 2 X 2 asymmetric orthogonal matrices is commutative.

III. SYMMETRIC BY ASYMMETRIC

$$S(\theta)A(\theta) = A(-\theta)S(\theta) = S(2\theta)$$

$$S(-\theta)A(-\theta) = A(\theta)S(-\theta) = S(\pi - \theta)A(\pi - \theta) = S(-2\theta)$$

$$A(\theta)S(\theta) = S(\theta)A(-\theta) = S(-\theta)A(\theta) = A(-\theta)S(-\theta) = A(\pi - \theta)S(\pi - \theta) = S(0)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S(\theta)A(\pi - \theta) = A(\pi - \theta)S(-\theta) = S(\pi - \theta)A(\theta) = A(-\theta)S(\pi - \theta) = S(-\pi) = S(\pi)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S(-\theta)A(\pi - \theta) = S(\pi - \theta)A(-\theta) = A(\theta)S(\pi - \theta) = S(\pi - 2\theta)$$

$$A(\pi - \theta)S(\theta) = S(2\theta - \pi)$$

In general, we have

$$S(\theta_1)A(\theta_2) = A(-\theta_2)S(\theta_1) = S(\theta_1 + \theta_2).$$

Thus, multiplication of 2 X 2 orthogonal matrices, one of which is symmetric and the other asymmetric, is not a commutative operation.

IV. SPECIAL MATRICES

Let $S = S(\theta)$ and $A = A(\theta)$

$S = S^3 = S^5 = \cdots = S^{2k+1} = \cdots$

$I = A(0) = S^2 = S^4 = S^6 = \cdots = S^{2k} = \cdots$

$A^{2k} = A(2k\theta) = A(\alpha)$ where $2k\theta = \alpha + 2\pi k$ and $0 \le \alpha < 2\pi$

$S(-\theta)S(\theta)\ S(-\theta)S(\theta) \cdots S(-\theta)S(\theta) = A(2\theta)A(2\theta) \cdots A(2\theta) = A(2n\theta) = A(\phi)$

where $2n\theta = \phi + 2\pi k$ and $0 \le \phi < 2\pi$.

$$S(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$$S(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$$

$$S(\pi - \theta) = \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = -S(-\theta)$$

$$S(\pi + \theta) = \begin{bmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = -S(\theta)$$

$$S(\pi/2 - \theta) = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

$$S(\pi/2 + \theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$S(3\pi/2 - \theta) = \begin{bmatrix} -\sin\theta & -\cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} = -S(\pi/2 - \theta)$$

$$S(3\pi/2 + \theta) = \begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} = -S(\pi/2 + \theta)$$

$$A(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A(\pi - \theta) = \begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix} = -A(-\theta)$$

$$A(\pi + \theta) = \begin{bmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = -A(\theta)$$

$$A(\pi/2 - \theta) = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

$$A(\pi/2 + \theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix}$$

$$A(3\pi/2 - \theta) = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} = -A(\pi/2 - \theta)$$

$$A(3\pi/2 + \theta) = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} = -A(\pi/2 + \theta)$$

V. OTHER EXAMPLES

$I_S^4(\theta^{2,3})I_S^4(\theta^{2,4})$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\theta & \sin\theta & \sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & \cos\theta & -\sin^2\theta \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

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